Inclusion (or subsethood) in type-2 fuzzy sets

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1. Abstract

Inspired by Kosko's work ([2]), we analyze the conditions needed to define an inclusion measure between type-1 fuzzy sets. These properties will be required by any inclusion measure between fuzzy sets or any of its extensions (interval-valued or type-2 fuzzy sets). We focus our work on type-2 fuzzy sets.
But in our paper the treatment of type-2 fuzzy sets differs from previous approaches. The degree of membership is not simply a fuzzy set as considered in other papers, but rather a label of the variable Truth.

2. Introduction

The definition of subsethood introduced by Zadeh in [6] is too strict. According to Zadeh's definition, it may occur that a fuzzy set is not a subset of another fuzzy set just because only one membership degree is greater. To relax this definitio Kosko introduced in [2] the degree of subsethood, or fuzzy subsethood, as a measure in fuzzy sets of the degree to which a fuzzy set is subset of another one.

Inspired by the Kosko's work, some authors axiomatize the properties of a measure of subsethood. Subsequently, several works are focused on adapting the definition of degree of subsethood to other extensions of fuzzy sets, such as interval type-2 fuzzy sets, or general type-2 fuzzy sets.

This contribution extends the definition of degree of subsethood, or fuzzy subsethood, to type-2 fuzzy sets from a novel perspective distinct from previous works.

3. Fuzzy sets and IVFSs

Definición 1 *(Zadeh [6]) A fuzzy set (FS) or type-1 fuzzy set (T1FS) in* X*,* A*, is characterized by a membership function* $\mu_A : X \to [0,1]$*, where* $\mu_A(x)$ *is the membership degree of an element* $x \in X$ *to the set* A*.*
A fuzzy set A *in* X *is subset of another fuzzy set* B*, denoted by* $A \subseteq B$*, if and only if* $\mu_A(x) \leq \mu_B(x)$ *for all* $x \in X$*.*

Definición 2 *(Zadeh [6]) An interval-valued fuzzy set (IVFS) in* X*,* $\mathbf{A}$*, is characterized by a membership function* $\sigma_{\mathbf{A}} : X \to I([0,1])$*, where* $I([0,1])$ *denotes the set of all closed subintervals of the unit interval* $[0,1]$*; that is* $[0,1] = \{[L,U] : 0 \leq L \leq U \leq 1\}$*.*
Let $\mathbf{A}$ *and* $\mathbf{B}$ *be IVFSs, with* $\sigma_A(x) = [L_{\mathbf{A}}(x), U_{\mathbf{A}}(x)]$ *and* $\sigma_{\mathbf{B}}(x) = [L_{\mathbf{B}}(x), U_{\mathbf{B}}(x)]$*. Then* $\mathbf{A}$ *is included in* $\mathbf{B}$*,* $\mathbf{A} \leq_I \mathbf{B}$*, if and only if for all* $x \in X$*,* $L_{\mathbf{A}}(x) \leq L_{\mathbf{B}}(x)$ *and* $U_{\mathbf{A}}(x) \leq U_{\mathbf{B}}(x)$*.*

4. Type-2 fuzzy sets

Definición 3 *(Mizumoto and Tanaka [1]) A type-2 fuzzy set (T2FS) in* X*,* $\mathcal{A}$*, is characterized by a membership function*

$$\mu_{\mathcal{A}} : X \to M = Map([0,1],[0,1]),$$

that is, $\mu_{\mathcal{A}}(x)$ *is a type-1 fuzzy set in the interval* $[0,1]$ *and also the membership degree of the element* $x \in X$ *in the set* $\mathcal{A}$*. Therefore,* $\mu_{\mathcal{A}}(x) = f_x$*, where* $f_x : [0,1] \to [0,1]$

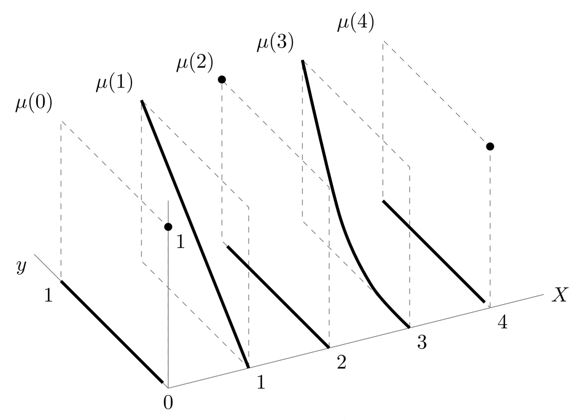

Fig 1. Example of a T2FS.

5. Subsethood in fuzzy sets and IVFSs

Definición 4 *(Young [5]) Let* A *and* B *be two fuzzy sets. A measure of subsethood degree for fuzzy sets, denoted by* $S(A,B)$*, is a mapping* $S : FS(X) \times FS(X) \to [0,1]$ *that satisfies the following properties:*

- *(S1)* $S(A,B) = 1$ *if and only if* $A \subseteq B$*.*
- *(S2) If* $P \subseteq A$*, then* $S(A, A^c) = 0$ *if and only if* $A = X$*, where* $P(x) = \frac{1}{2}$ *for all* $x \in X$*.*
- *(S3) If* $B \subseteq A_1 \subseteq A : 2$*, then* $S(A_2, B) \leq (A_1, B)$ *and if* $B_1 \subseteq B_2$*, then* $S(A, B_1) \leq (A, B_2)$*.*

Definición 5 *(Vlachos and Sergiadis [3]) Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two interval-valued fuzzy sets. A measure of subsethood degree for IVFS, is a mapping* $\mathbf{S} : IVFS(X) \times IVFS(X) \longrightarrow [0,1]$ *that satisfies the following properties:*

- $(\mathbf{S1})$ $\mathbf{S}(\mathbf{A},\mathbf{B}) = 1$ *if and only if* $\mathbf{A} \leq_I \mathbf{B}$*.*
- $(\mathbf{S2})$ *If* $\mathbf{A}^c \leq_I \mathbf{A}$*, then* $\mathbf{S}(\mathbf{A}, \mathbf{A}^c) = 0$ *if and only if* $\mu_{\mathbf{A}}(x) = [1,1]$ *for all* $x \in X$*.*
- $(\mathbf{S3})$ *If* $\mathbf{B} \leq_1 \mathbf{A}_1 \leq_1 \mathbf{A}_2$*, then* $\mathbf{S}(\mathbf{A}_2, \mathbf{B}) \leq (\mathbf{A}_1, \mathbf{B})$ *and if* $\mathbf{B}_1 \leq_1 \mathbf{B}_2$*, then* $\mathbf{S}(\mathbf{A}, \mathbf{B}_1) \leq (\mathbf{A}, \mathbf{B}_2)$

6. Previous definitions

Definición 6 *(Walker and Walker [4]) Let* $f \in [0,1]^{[0,1]}$*, we define* $f^L(x) = sup\{f(y); y \leq x\}$ *and ,* $f^R(x) = sup\{f(y); y \geq x\}$

Teorema 7 *Let* $f, g \in [0,1]^{[0,1]}$ *normal and convex. Then* $f \sqsubseteq g$ *if and only if* $f^L(x) \geq g^L(x)$ *and* $f^R(x) \leq g^R(x)$*.*

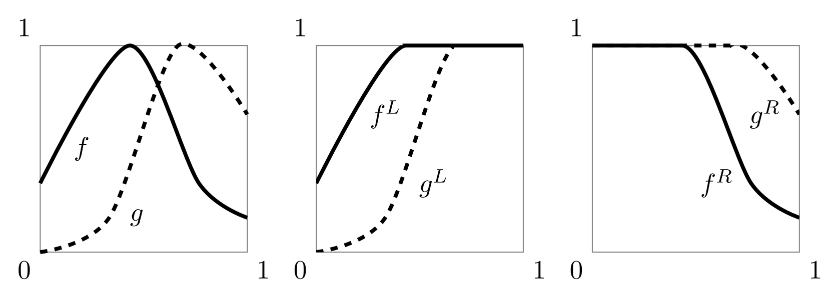

Fig 2. $f, g \in [0,1]^{[0,1]}$ normal and convex with $f \sqsubseteq g$.

Definición 8 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *T2FSs, with* $\mu_{\mathcal{A}}(x) = f_x$ *and* $\mu_{\mathbf{B}}(x) = g_x$ *for all* $x \in X$*. Then* $\mathbf{A} \subseteq^* \mathbf{B}$ *if and only if for all* $x \in X$*,* $(f^L(x)(y)) \geq (g^L(x)(y))$ *and* $(f^R(x)(y)) \leq (g^R(x)(y))$*, except in a set of measure zero.*

7. Subsethood in T2FS

Definición 9 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two T2FSs whit the values of the membership functions* $\mu_{\mathcal{A}}(x)$ *and* $\mu_{\mathcal{B}}(x)$*, respectively, being normal and convex functions for all* $x \in X$*. A function* $\mathcal{I} : T2FSs(X) \times T2FSs(X) \to [0,1]$ *is an inclusion or subsethood measure in* $T2FSs(X)$ *if*

- *(I1)* $\mathcal{I}(\mathcal{A}, 0) = 0$ *if* $\mathcal{A} \neq 0$*, being* $\prime$ *the empty T2FS, that is,* $0(x) = \overline{0}$ *for all* $x \in X$*, where* $\overline{0}(0) = 1$ *and* $\overline{0}(y) = 0$ *for all* $y \neq 0$*.*
- *(I2)* $\mathcal{I}(\mathcal{A}, \mathcal{B}) = 1 \Longleftrightarrow \mathcal{A} \subseteq^* \mathcal{B}$*.*
- *(I3)* $\mathcal{B} \subseteq^* \mathcal{C} \Longrightarrow \mathcal{I}(\mathcal{A}, \mathcal{B}) \leq \mathcal{I}(\mathcal{A}, \mathcal{C})$*.*
- *(I4)* $\mathcal{A} \subseteq^* \mathcal{B} \subseteq^* \mathcal{C} \Longrightarrow \mathcal{I}(\mathcal{C}, \mathcal{A}) \leq \mathcal{I}(\mathcal{B}, \mathcal{A})$ *and* $\mathcal{I}(\mathcal{C}, \mathcal{A}) \leq \mathcal{I}(\mathcal{C}, \mathcal{B})$*.*
- *(I5) Let* $\mathcal{A}^c \subseteq^* \mathcal{A}$*. Then* $\mathcal{I}(\mathcal{A}, \mathcal{A}^c) = 0 \Longleftrightarrow \mathcal{A} = \mathcal{X}$*, being* $\mathcal{X}$ *the universal T2FS, that is,* $\mathcal{X}(x) = \overline{1}$ *for all* $x \in X$*, where* $\overline{1}(1) = 1$ *and* $\overline{1}(y) = 0$ *for all* $y \neq 1$*.*

8. An inclusion measure in T2FS

Among others, we have the following inclusion measure in T2FSs(X), $\mathcal{I}$, where the cardinal of X is N.
$\mathcal{I}(\mathcal{A}, \mathcal{B}) = 1$ if $\mathcal{A} = 0$

$$\text{Elsewhere } \mathcal{I}(\mathcal{A}, \mathcal{B}) = \frac{\sum_{i=1}^{N}\left(1 - \int_0^1 max(\mu_A(x_i)^L(y), \mu_B(x_i)^L(y))dy + \int_0^1 min(\mu_A(x_i)^R(y), \mu_B(x_i)^R(y))dy\right)}{\sum_{i=1}^{N}\left(1 - \int_0^1 \mu_A(x_i)^L(y)dy + \int_0^1 \mu_A(x_i)^R(y)dy\right)}$$

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