Subsethood in type-2 fuzzy sets

Carmen Torres-Blanc, Luis Magdalena, Susana Cubillo and Jesus Martinez-Mateo

Departamento de Matemática Aplicada a las Tecnologías de la Información y las Comunicaciones, Universidad Politécnica de Madrid, Spain.

Abstract. The definition of *subsethood* introduced by Zadeh [5] is rarely rigorous, in a strict sense, in the context of fuzzy set theory. According to Zadeh's definition, it may occur that a fuzzy set A is not a subset of another fuzzy set B just because only one membership degree in A is greater than in B, that is, $\mu_A(x) > \mu_B(x)$ for a single element x of the Universe X.
Instead, Kosko introduced in [2] the degree of subsethood, or *fuzzy subsethood*, denoted by $S_K(A, B)$, as a measure in fuzzy sets of the degree to which a fuzzy set A is subset of another fuzzy set B. He proposes as such a measure a normalized sum of violations in the membership degree, that is, considering magnitude and proportion of these violations as follows.
Surprisingly, Kosko realizes that the degree of subsethood reduces to cardinalities, that is, $S_K(A, B) = |A \cap B|/|A|$, but he also realizes that the degree of subsethood looks like and behave as a conditional probability (fuzzy conditioning), and thus cardinality can be interpreted as probabilities, being $|A \cap B|$ and $|A|$ the joint and marginal probabilities, respectively. Finally, he also shows that fuzzy entropy reduces to fuzzy conditioning, that is, to a subsethood degree measuring. The entropy of a fuzzy set A is the degree to which the union set between A and its complementary fuzzy set A^c is a subset of the intersection set between A and its complementary A^c, that is, $H_K(A) = S_K(A \cup A^c, A \cap A^c)$. Next, he proves that this is a nonprobabilistic fuzzy entropy since it satisfies the axiomatic properties of entropy proposed by De Luca and Termini [1] for fuzzy sets.
Later, Young considered in [4] a number of axiomatic properties for the measure of degree of subsethood introduced by Kosko. Then she proves that any measure of subsethood degree, $S(A, B)$, that satisfies these axioms provides a measure of fuzzy entropy according to the previous definition of entropy by Kosko, that is, $S(A \cup A^c, A \cap A^c)$ is a fuzzy entropy.
Finally, Vlachos and Sergiadis [3], inspired by the Kosko's and Young's works, extended the definition of degree of subsethood to interval-valued fuzzy sets. They also propose a set of axioms required by any measure of subsethood in interval-valued fuzzy sets, and they prove that any subsethood measure satisfying this set of axioms also produces an entropy, as the one described by Kosko.
We have extended the definition of degree of subsethood to type-2 fuzzy sets. In order to define a subsethood in these sets we firstly introduce an order. Next, we define the union, intersection, complementary and cardinality in type-2 fuzzy sets. To conclude with a set of axioms required

by any subsethood measure and some measures of subsethood degree in type-2 fuzzy sets. And, as in Vlachos' work, we study if any subsethood measure satisfying this set of axioms also produce an entropy.

Keywords: Subsethood · Entropy · Type-2 fuzzy set.

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