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Subsethood measures based on cardinality of type-2 fuzzy sets

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ABSTRACT

In this work we present a novel axiomatic framework for subsethood measures in type-2 fuzzy sets. It differs from previous approaches in two key ways. First, the degree of membership is not simply a fuzzy set as considered in other papers, but rather a label of the variable *Truth*, more in line with Zadeh's original idea. Secondly, the concept of subsethood is approached in terms of its relationship with cardinality. Additionally, illustrative examples of such measures are provided.

1. Introduction

The definition of *subsethood* introduced by Zadeh in [1] is rarely rigorous, in a strict sense, in the context of fuzzy set theory. According to Zadeh's definition, it may occur that a fuzzy set is not a subset of another fuzzy set just because only one membership degree is greater. To relax this definition Kosko introduced in [2] the degree of subsethood, or *fuzzy subsethood*, as a measure in fuzzy sets of the degree to which a fuzzy set is subset of another one. Surprisingly, Kosko realizes that the proposed measure of subsethood degree reduces to cardinalities. Furthermore, he is able to show that this subsethood degree measure behaves as a conditional probability, that is, *fuzzy conditioning*.

However, the aim of Kosko's work was to introduce a new entropy measure satisfying the axiomatic properties for a nonprobabilistic fuzzy entropy proposed by De Luca and Termini in [3]. Hence, Kosko also proposes a fuzzy entropy measure that novelly reduces to fuzzy conditioning, that is, to a subsethood degree measuring.

Inspired by the Kosko's work, Sinha and Dougherty [4] considered axiomatizing the properties of a measure of subsethood. The authors identified a total of up to twelve axioms (although the last three are considered additional properties) that any measure of fuzzy subsethood must comply with. Subsequently, Young [5] reduced the number of axioms to three, and proved that a measure of fuzzy subsethood that satisfies only those three axioms reduces to a fuzzy entropy measure, such as that defined by Kosko. A comprehensive overview of the various existing axiomatics in the literature for fuzzy subsethood measures and their interrelations can be found in [6,7].

Later, Vlachos and Sergiadis [8] extended the definition of degree of subsethood to interval-valued fuzzy sets. Building upon the work of Kosko and Young, they proposed a set of axioms that any measure of subsethood in interval-valued fuzzy sets should satisfy.

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They proved that any fuzzy subsethood measure satisfying this set of axioms also produces an entropy, again, such as the one defined by Kosko.

Subsequently, several works are focused on adapting the definition of degree of subsethood to other extensions of fuzzy sets, such as interval type-2 fuzzy sets [9], or general type-2 fuzzy sets [10]. In this last paper, Takáč provides a method to obtain subsethood measures using the α-plane representation for type-2 fuzzy sets, but he stops short of proposing any subsethood measure. Other works focus also on the description of new methods and efficient algorithms for constructing fuzzy subsethood measures [11–13].

Zadeh highlighted (see [14,15]) that experts may encounter difficulties in determining the precise extent to which an element belongs to a set or verifies a property. Furthermore, it is more realistic to assume that experts express their knowledge in natural language, utilising expressions such as "approximately", "about", "almost never", and so forth. Consequently, Zadeh identified a requirement to develop a novel conceptualisation of fuzzy logic or fuzzy sets, wherein the membership or truth degrees can be represented by linguistic labels, as opposed to numerical values within the range [0,1] or closed intervals within this range. Accordingly, Zadeh's seminal work gave rise to the concept of type-2 fuzzy sets, which represent an extension of both fuzzy sets and interval-valued sets. In this approach, each membership degree is represented by a label denoting the variable 'truth'. This concept was subsequently revisited and structured by Mizumoto in reference [29]. Our work is based on Zadeh's original idea of type-2 sets.

In addition, type-2 fuzzy sets have been extensively studied during the recent years. This is evidenced by the great amount of works dedicated to both the theoretical framework (for example [16–20]) and the applications (see, for instance [21–26]). These applications can be found in many areas of artificial intelligence such as control (see [22–24]), image processing (see, for instance [25]) or deep learning (see [26]). It is therefore important to continue to develop models that improve reasoning and interpretation with these types of fuzzy sets.

The main contributions of this manuscript are as follows. On the one hand, the axiomatic definitions of cardinality and subsethood measures for type-2 fuzzy sets, consistent with the corresponding axioms for interval-valued fuzzy sets. Various cardinality measures that satisfy these axioms are given and used to define subsethood measures in a similar way to Kosko for type-1 fuzzy sets. On the other hand, another contribution is the treatment of membership degrees of type-2 fuzzy sets. As already mentioned, we treat type-2 fuzzy sets as originally conceived by Zadeh and Mizumoto, where, given a universe X, the degree of membership of an element x of the universe to a type-2 fuzzy set A, namely $\mu_A(x)$, is a linguistic label. However, most authors treat type-2 fuzzy sets as type-1 fuzzy sets over the universe $X \times [0,1]$, and this is not an extension of type-1 fuzzy sets, nor of interval-valued fuzzy sets. Consequently, we believe that type-2 fuzzy sets should be treated as originally given by Zadeh and Mizumoto. To the best of our knowledge, this is the first time, except for Takáč's work, that subset measures for type-2 fuzzy sets are studied by treating membership degrees as linguistic labels. Moreover, as we have already pointed out, Takáč did not give any examples of subset measures for type-2 fuzzy sets.

The remainder of this paper is organized as follows. In Section 2 we introduce fuzzy sets and their extensions, we analyze several orders for intervals and interval-valued fuzzy sets, and define some basic operations needed throughout the paper. Section 3 contains a discussion about the cardinality axioms required for fuzzy sets, interval-valued fuzzy sets and type-2 fuzzy sets. Next, in Section 4 we introduce the degree of subsethood in fuzzy and interval-valued fuzzy sets and the axiomatic requirements for a fuzzy subsethood measure inspired by Kosko, Young and Vlachos' works. In Section 5 we extend the definition of fuzzy subsethood to type-2 fuzzy sets, and propose different subsethood measures for these sets. Finally, we present our concluding remarks in Section 6.

2. Fuzzy sets and their extensions

Throughout this paper X denotes a finite non-empty set called universe of discourse, and $\leq$ denotes the usual order relation (less than or equal to) such that the set of real numbers is a lattice with respect to this order. In this section, different extensions of fuzzy sets will be presented, starting with Zadeh's definition of fuzzy set.

Definition 2.1. (Zadeh [1]) A fuzzy set (FS) or type-1 fuzzy set (T1FS), A, is characterized by a membership function μ_A,

$$\mu_A : X \to [0,1],$$

where $\mu_A(x)$ is the membership degree of an element $x \in X$ in the set A.

Commonly, we refer to a fuzzy set A using its membership function μ_A. We denote by $FS(X)$, or equivalently by $[0,1]^X = \mathrm{Map}(X,[0,1])$, the set of all (type-1) fuzzy sets on X. From the usual order $\leq$ in $[0,1]$ we define a partial order on $FS(X)$ as follows: $\mu_A \leq \mu_B$ if and only if $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$. Therefore, $([0,1]^X, \leq)$ is a bounded lattice with smallest and greatest elements the constant functions $\mathbf{0}$ and $\mathbf{1}$, respectively, that is, $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in X$. This order ensures that the membership degrees of an element in two arbitrary fuzzy sets, A and B, are always comparable using the usual order in $[0,1]$. This implies that one of the elements is equal to or greater than the other. In accordance with this, Zadeh proposed in [1] the following definition of subsethood in fuzzy sets.

Definition 2.2. (Zadeh [1]) A fuzzy set A is subset of another fuzzy set B, denoted by $A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$, $\forall x \in X$.

The basic Zadeh's operations for fuzzy sets are the following.

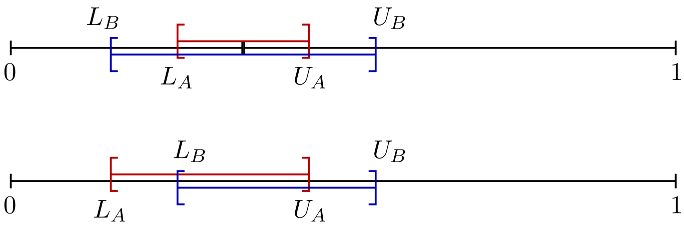

Fig. 1. Orders in intervals. Top shows the interval order for the embedding relationship "**A** is contained in **B**", while bottom shows the interval order for the subsethood relationship "**A** is lower than or equal to **B**".

Definition 2.3. The union between A and B, denoted by $A \cup B$, is given by

$$\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x)) \quad \forall x \in X.$$

The intersection between A and B, denoted by $A \cap B$, is given by

$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x)) \quad \forall x \in X.$$

The complementary set of A, denoted by A^c, is given by

$$\mu_{A^c(x)} = 1 - \mu_A(x) \quad \forall x \in X.$$

Furthermore, Zadeh proposed novel extensions to the concept of fuzzy sets in his works [14,15]. One of these extensions, called type-2 fuzzy sets, is defined in Definition 2.10. For a comprehensive examination and motivation of the various extensions of fuzzy sets, we direct the reader to the article [27]. However, it is first necessary to define interval-valued fuzzy sets, which are specific cases of type-2 fuzzy sets. This is a consequence of the fact that some of the concepts presented here generalize terms that were previously established for these kinds of sets.

Definition 2.4. (Zadeh [14]) An interval-valued fuzzy set (IVFS), $\mathbf{A}$, is characterized by a membership function $\sigma_{\mathbf{A}}$,

$$\sigma_{\mathbf{A}} : X \to I([0,1]),$$

where $I([0,1])$ denotes the set of all closed subintervals of the unit interval $[0,1]$, that is, $I([0,1]) = \{[L,U] : 0 \le L \le U \le 1\}$.

The membership degree of an element $x \in X$ in the set $\mathbf{A}$ is then a closed subinterval in $I([0,1])$, that is, $\sigma_{\mathbf{A}}(x) = [L_{\mathbf{A}}, U_{\mathbf{A}}]$, where $L_{\mathbf{A}}$ and $U_{\mathbf{A}}$ are the lower and upper membership degrees, respectively. Again, we commonly refer to an interval-valued fuzzy set $\mathbf{A}$ using its membership function $\sigma_{\mathbf{A}}$. We denote by $IVFS(X)$, or equivalently by $I([0,1])^X$, the set of all interval-valued fuzzy sets on X.

Orders considered in IVFSs are typically defined from orders on intervals. The present paper will consider only two distinct orders when employed to define orders for IVFSs. Let $A, B \in I([0,1])$ be such that $A = [L_A, U_A]$ and $B = [L_B, U_B]$. We define such orders as follows.

Definition 2.5 *(Lattice order).* We say that A is lower than or equal to B, and denoted by $A \le_I B$, if and only if $L_A \le L_B$ and $U_A \le U_B$ (see bottom of Fig. 1).

Definition 2.6 *(Contained order).* We say that A is contained in B, and denoted by $A \subseteq B$, if and only if $L_B \le L_A \le U_A \le U_B$ (see top of Fig. 1).

Note that $(I([0,1]), \le_I)$ is a bounded lattice with smallest and greatest elements $0_I = [0,0]$ and $1_I = [1,1]$, respectively. However, $(I([0,1]), \subseteq)$ is not a lattice since there are pairs of intervals without an infimum. For instance, those intervals whose intersection is the empty set have no infimum since the empty set is not a closed interval.

We will now examine the role of these two interval orders in IVFSs. Let $\mathbf{A}$ and $\mathbf{B}$ be two IVFSs, with $\sigma_{\mathbf{A}}$ and $\sigma_{\mathbf{B}}$ denoting their respective membership functions. Consequently, for each $x \in X$, the values $\sigma_{\mathbf{A}}(x)$ and $\sigma_{\mathbf{B}}(x)$ represent intervals. Two relations (partial orders) can be defined in IVFSs, derived from the previously defined contained and lattice orders. The first one is called subsethood:

Definition 2.7 *(Subsethood).* $\mathbf{A}$ is included in (is a subset of) $\mathbf{B}$, and it is denoted by $\mathbf{A} \le_I \mathbf{B}$, if and only if $\sigma_{\mathbf{A}}(x) \le_I \sigma_{\mathbf{B}}(x), \forall x \in X$.

Clearly, $(I([0,1])^X, \le_I)$ is a bounded lattice with smallest and greatest elements σ_{0_I} and σ_{1_I}, respectively, with $\sigma_{0_I}(x) = 0_I = [0,0]$ and $\sigma_{1_I}(x) = 1_I = [1,1]$ for all $x \in X$. According the order $\le_I$, we can define the following widely studied operations:

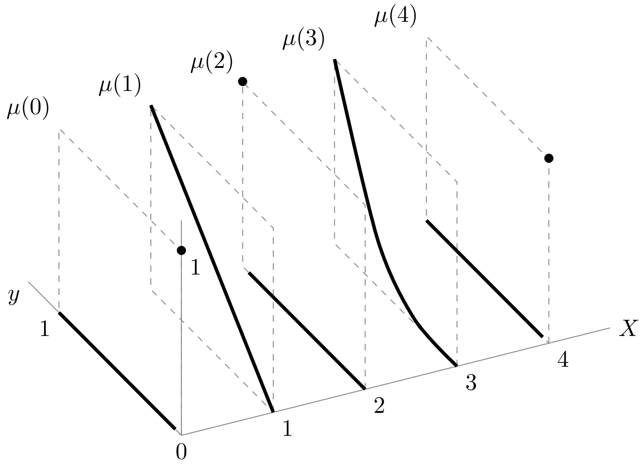

Fig. 2. Example of a T2FS.

Definition 2.8. The union between **A** and **B**, denoted by $\mathbf{A} \cup \mathbf{B}$, is given by

$$(\sigma_{\mathbf{A}\cup\mathbf{B}})(x) = [\max(L_{\mathbf{A}}(x), L_{\mathbf{B}}(x)), \max(U_{\mathbf{A}}(x), U_{\mathbf{B}}(x))], \quad \forall x \in X.$$

The intersection between **A** and **B**, denoted by $\mathbf{A} \cap \mathbf{B}$, is given by

$$(\sigma_{\mathbf{A}\cap\mathbf{B}})(x) = [\min(L_{\mathbf{A}}(x), L_{\mathbf{B}}(x)), \min(U_{\mathbf{A}}(x), U_{\mathbf{B}}(x))], \quad \forall x \in X.$$

The complementary set of **A**, denoted by $\mathbf{A}^c$, is given by

$$\sigma_{\mathbf{A}^c}(x) = [1 - U_{\mathbf{A}}(x), 1 - L_{\mathbf{A}}(x)], \quad \forall x \in X.$$

The second order relation on IVFSs that we are going to discuss is embedding.

Definition 2.9 *(Embedding)*. **A** is embedded in **B**, and it is denoted by $\mathbf{A} \subseteq \mathbf{B}$, if and only if $\sigma_{\mathbf{A}}(x) \subseteq \sigma_{\mathbf{B}}(x), \forall x \in X$.

The meaning of both order relations differs. Let **A** and **B** be two IVFSs. On the one hand, with the subsethood relation we can state that the information given by **A**, when **A** is included in **B**, is *less true* than the information given by **B**. On the other hand, with the embedding relation we can state that the information given by **A**, when **A** is embedded in **B**, is *more precise* about the real membership function than **B**. For a more exhaustive analysis of embedding in IVFSs see [28].

It is evident that both interpretations, the subsethood and the embedding, are applicable in IVFSs but not in fuzzy sets. We will focus on the subsethood concept, as it appears repeatedly in fuzzy set theory. As an example, it has been extensively used for defining entropy in fuzzy sets (see [2,7,5]). The concept of subsethood can be defined for type-2 fuzzy sets with a similar interpretation as that of IVFSs. First, let us define what a type-2 fuzzy set is.

Definition 2.10. (Mizumoto and Tanaka [29,30]) A type-2 fuzzy set (T2FS), $\mathcal{A}$, is characterized by a membership function

$$\mu_{\mathcal{A}} : X \to \mathbf{M} = Map([0,1],[0,1]),$$

that is, $\mu_{\mathcal{A}}(x)$ is a type-1 fuzzy set in the interval $[0,1]$ and also the membership degree of the element $x \in X$ in the set $\mathcal{A}$. Therefore, $\mu_{\mathcal{A}}(x) = f_x$ where $f_x : [0,1] \to [0,1]$.

We denote by $T2FS(X) = Map(X, \mathbf{M})$ the set of all type-2 fuzzy sets on X. Fig. 2 shows an example of a type-2 fuzzy set on the finite Universe of discourse $X = \{0,1,2,3,4\}$ and a membership function $\mu : X \to \mathbf{M}$.

It should be noted that the membership degree of an element $x \in X$ in T2FSs is a function $f \in \mathbf{M}$ that can be perceived as a label of the variable *truth*, as originally interpreted by Zadeh in [14,15]. Furthermore, these labels are usually normal and convex, that is, $\sup\{f(x) : x \in [0,1]\} = 1$ and for any $x \le y \le z$ it holds that $f(y) \ge f(x) \wedge f(z)$, respectively. In this work we only consider this type of labels, and we denote the set of all such labels by:

$$\mathcal{L} = \{f \in \mathbf{M} : f \text{ normal and convex}\}. \tag{1}$$

The reason to impose such restriction is that in $\mathcal{L}$ we are able to define an order $\sqsubseteq$ such that $(\mathcal{L}, \sqsubseteq)$ is a complete lattice. In order to gain a deeper understanding of the structure of this lattice, it is necessary to provide some preliminary concepts and definitions. Let us begin by introducing some auxiliary functions, outlining their properties and illustrating their significance.

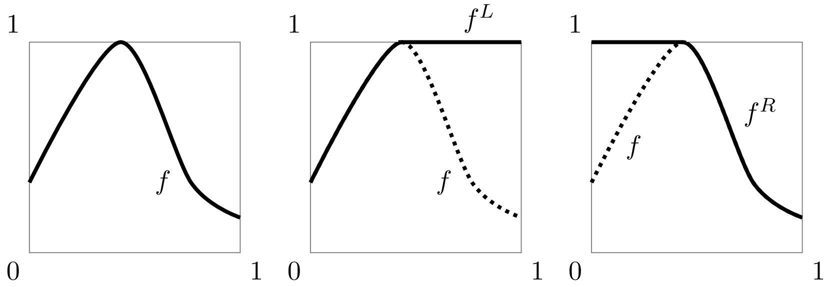

Fig. 3. Example of f^L and f^R.

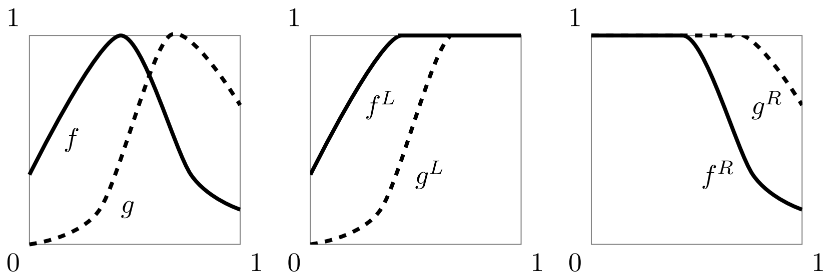

Fig. 4. Example where $f \sqsubseteq g$.

Definition 2.11. (Walker and Walker [31]) Let $f \in [0,1]^{[0,1]}$, we define $f^L, f^R \in [0,1]^{[0,1]}$ as follows:

$$f^L(x) = \sup\{f(y) : y \le x\}, \quad f^R(x) = \sup\{f(y) : y \ge x\}.$$

Note that, f^L and f^R are monotonically increasing and decreasing, respectively. Moreover, $f \le f^L$, $f \le f^R$, $(f^L)^L = f^L$ and $(f^R)^R = f^R$, for all $f \in \mathbf{M}$, where $\le$ is the usual order in the set of functions (that is, $f \le g$ if and only if $f(x) \le g(x)$, $\forall x \in [0,1]$). An example of both f^L and f^R is shown in Fig. 3. The following characterization of the partial order $\sqsubseteq$ on $\mathcal{L}$ in terms of f^L and f^R was given in [32] (see Fig. 4).

Theorem 2.12. *(Harding et al. [32]) Let $f, g \in \mathcal{L}$. Then, $f \sqsubseteq g$ if and only if $g^L \le f^L$ and $f^R \le g^R$.*

For the sake of simplicity, we will utilise this characterisation as our definition. See [32,31], for more information about the properties and definition of this order.

The next definitions of functions in $\mathcal{L}$ play an important role in the subsequent discussion. Indeed, the greatest and smallest elements in $(\mathcal{L}, \sqsubseteq)$ are functions satisfying Definition 2.13.

Definition 2.13. (Walker and Walker [31]) Let $a \in X$. The characteristic function of a is $\bar{a} : X \to [0,1]$, where

$$\bar{a}(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \ne a. \end{cases}$$

Definition 2.14. (Walker and Walker [31]) Let $[a,b] \subseteq [0,1]$. The characteristic function of $[a,b]$ is $\overline{[a,b]} : [0,1] \to [0,1]$, where

$$\overline{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{if } x \notin [a,b]. \end{cases}$$

It can be checked in [32,29–31] that $(\mathcal{L}, \sqsubseteq, \bar{0}, \bar{1})$ is a complete lattice where $\bar{0}$ is the smallest element and $\bar{1}$ is the greatest element. Based on this order we have the following operations on $\mathcal{L}$ in terms of the minimum ($\wedge$) and the maximum ($\vee$) of two real numbers.

Definition 2.15. ([32], [31]) Let $f, g \in \mathcal{L}$. The union operator $\sqcup$ (extended maximum) between f and g is given by

$$(f \sqcup g)(x) = \sup\{f(y) \wedge g(z) : y \vee z = x\}.$$

The intersection operator $\sqcap$ (extended minimum) between f and g is given by

$$(f \sqcap g)(x) = \sup\{f(y) \wedge g(z) : y \wedge z = x\}.$$

The complementary of f, denoted by $\neg f$, is given by

$$(\neg f)(x) = \sup\{f(y) : 1 - y = x\} = f(1 - x).$$

Remark 2.16. As stated in [31], for the given order $\sqsubseteq$ in $\mathcal{L}$ we have the next properties:

- $f \sqcup g = g$ if and only if $f \sqsubseteq g$,
- $f \sqcap g = f$ if and only if $f \sqsubseteq g$,

and thus:

- $f \sqcap \overline{0} = \overline{0}$ and $f \sqcup \overline{0} = f$.
- $f \sqcap \overline{1} = f$ and $f \sqcup \overline{1} = \overline{1}$.

With the previous operators in $\mathcal{L}$ we can define the next operations on $\mathcal{L}^X = Map(X, \mathcal{L})$ between T2FSs.

Definition 2.17. Given $\mathcal{A}, \mathcal{B} \in \mathcal{L}^X$. The union between $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \sqcup \mathcal{B}$, is given by:

$$\mu_{\mathcal{A}\sqcup\mathcal{B}}(x) = \mu_{\mathcal{A}}(x) \sqcup \mu_{\mathcal{B}}(x), \quad \forall x \in X.$$

The intersection between $\mathcal{A}$ and $\mathcal{B}$, denoted by $\mathcal{A} \sqcap \mathcal{B}$, is given by:

$$\mu_{\mathcal{A}\sqcap\mathcal{B}}(x) = \mu_{\mathcal{A}}(x) \sqcap \mu_{\mathcal{B}}(x), \quad \forall x \in X.$$

The complementary set of $\mathcal{A}$, denoted by $\mathcal{A}^c$, is given by:

$$\mu_{\mathcal{A}^c}(x) = \neg\mu_{\mathcal{A}}(x), \quad \forall x \in X.$$

Finally note that any (type-1) fuzzy set can also be defined as an IVFSs. Clearly, any membership function $\mu : X \to [0, 1]$ corresponds to a membership function:

$$\begin{aligned} \sigma : \; & X \to I([0,1]) \\ & x \mapsto [\mu(x), \mu(x)] \end{aligned}$$

Analogously, any IVFS can also be defined as a T2FS. Indeed, any membership function $\sigma : X \to I([0, 1])$ corresponds to a membership function:

$$\begin{aligned} \varphi : \; & X \to \mathcal{L} \subset \mathbf{M} \\ & x \mapsto \overline{\sigma(x)} \end{aligned}$$

Therefore, we can state that T2FS extend IVFS, and IVFS extend FS (and obviously, T2FS extend FS). Note also that for any two intervals $[a_1, b_1], [a_2, b_2]$ we have that:

$$[a_1, b_1] \leq_I [a_2, b_2] \text{ if and only if } \overline{[a_1, b_1]} \sqsubseteq \overline{[a_2, b_2]}.$$

Hence, the order $\sqsubseteq$ in T2FSs is consistent with $\leq_I$ in IVFSs. This is one of the reasons why we strongly recommend the use of this order, despite the fact that some authors have been using the usual order in functions with two variables to compare two T2FSs $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{A} \leq \mathcal{B} \text{ if and only if } \mu_A(x)(y) \leq \mu_B(x)(y) \text{ for all } x \in X \text{ and for all } y \in [0, 1].$$

The papers [33–35] are an example. However, this order does not generalize the lattice order of IVFSs. In fact, not even the element that is supposed to be the smallest is lower than the one that is supposed to be the largest, since $\overline{0} \nleq \overline{1}$. In addition, the aforementioned works cite Mizumoto and Tanaka in [29,30] as the source where this order was first defined. However, the order defined by Mizumoto and Tanaka is precisely $\sqsubseteq$ and not $\leq$. Thus, the operators presented in Section 5 are going to measure the subsethood degree of two T2FSs in relation to the order $\sqsubseteq$. If our goal were to define embedding measures for T2FSs, we would use the $\leq$ order, since it generalizes the concept of embedding for IVFSs. Furthermore, if $\mathcal{A} \leq \mathcal{B}$, the information given by $\mathcal{A}$ is *more precise* than that given by $\mathcal{B}$. Therefore, previous works devoted to defining subsethood measures, such as [33–36], are actually dealing with embedding measures.

3. Cardinality of fuzzy sets, interval-valued fuzzy sets and type-2 fuzzy sets

Cardinality offers a so fuzzy response to the question of the number of elements within a given set X that belong to a particular fuzzy set. It is greater the more degree of membership each element of X has within the fuzzy set. Consequently, cardinality operators measure the size of fuzzy sets and provide a way to compare that size. The study of cardinality begins with the definition of the sigma-count of fuzzy sets in [3]. This first cardinality was used to define entropy in fuzzy sets. As we will see in Section 4, various measures of cardinality have been employed to define subsethood measures for a number of extensions of fuzzy sets. An axiomatic approach to the desired characteristics of a cardinality (also referred to as scalar cardinality) can be found in [37]. This text provides a comprehensive overview of previous approaches and justifies the following definition.

Definition 3.1 *([37]).* A function $sc : FS(X) \to [0, \infty)$ will be called scalar cardinality of FSs if the following properties are satisfied:

1. $sc(1/x) = 1$ for all $x \in X$.
2. For all $a, b \in [0,1]$ and $x, y \in X$, if $a \leq b$ then $sc(a/x) \leq sc(b/y)$.
3. For all $A, B \in [0,1]^X$, if $Supp(A) \cap Supp(B) = \emptyset$ then

$$sc(A \cup B) = sc(A) + sc(B).$$

Remark 3.2. Note that $a/x : X \to [0,1]$ is a fuzzy set such that:

$$(a/x)(y) = \begin{cases} a & \text{if } y = x, \\ 0 & \text{if } y \neq x \end{cases}$$

and that $Supp(A) = \{x \in X \mid \mu_A(x) > 0\}$.

The following result gives a characterization for the scalar cardinality and a method to build specific examples.

Theorem 3.3 *([37]). A mapping $sc : FS(X) \to [0, \infty)$ is a scalar cardinality of FSs if and only if there exists a monotonically increasing function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$ such that:*

$$sc(A) = \sum_{x \in X} f(\mu_A(x))$$

for each fuzzy set A.

Example 3.4. The sigma-count $|A| = \sum_{x \in X} \mu_A(x)$ of a fuzzy set A is a scalar cardinality.

Deschrijver and Král extended the definition of scalar cardinality to IVFSs (see [38]). However, they defined them as functions from $IVFS(X)$ to $\{[a,b] \subset \mathbb{R}^+ \mid 0 \leq a \leq b\}$. We think it is more suitable to use cardinalities where the image is a positive real number. Consequently, we prefer the next definition given by Hamrawi and Coupland in [39].

Definition 3.5 *([39]).* A function $sc : IVFS(X) \to [0, \infty)$ will be called scalar cardinality of IVFSs if the following properties are satisfied:

1. If $\mathbf{A}$ is a crisp set then $sc(\mathbf{A}) = |Supp(\mathbf{A})|$.
2. Given $\mathbf{A}, \mathbf{B} \in IVFS(X)$, if $\mathbf{A} \leq_I \mathbf{B}$ then $sc(\mathbf{A}) \leq sc(\mathbf{B})$.
3. For all $\mathbf{A}, \mathbf{B} \in IVFS(X)$, if $Supp(\mathbf{A}) \cap Supp(\mathbf{B}) = \emptyset$ then

$$sc(\mathbf{A} \cup \mathbf{B}) = sc(\mathbf{A}) + sc(\mathbf{B}).$$

Remark 3.6. Note that an $IVFS$ $\mathbf{A}$ is called crisp if $\sigma_{\mathbf{A}}(x) \in \{[0,0],[1,1]\}$ for all $x \in X$. We define the support of $\mathbf{A}$ as $Supp(\mathbf{A}) = \{x \in X \mid \sigma_{\mathbf{A}}(x) \neq [0,0]\}$.

In [8], Vlachos and Sergiadis proposed the following cardinality for an IVFS $\mathbf{A}$:

$$|\mathbf{A}| = \sum_{x \in X} \frac{L_{\mathbf{A}}(x) + U_{\mathbf{A}}(x)}{2}. \tag{2}$$

This represents the sum of the centroids of all intervals in the image of $\mathbf{A}$, and thus the cardinality can be defined as the average membership degree of each interval. This definition for cardinality is consistent with all the axioms set out in Definition 3.5. However, in a more general approach, the cardinality in IVFSs may be defined using a parameter α, with $0 \leq \alpha \leq 1$, as follows:

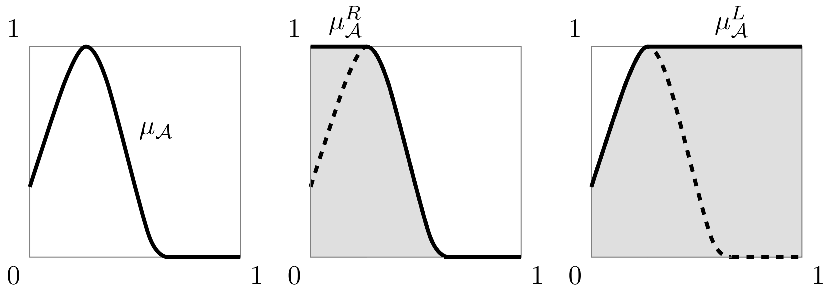

Fig. 5. Areas under the curves $\mu^R_{\mathcal{A}}$ and $\mu^L_{\mathcal{A}}$.

$$|\mathbf{A}|_\alpha = \sum_{x\in X} (\alpha L_{\mathbf{A}}(x) + (1-\alpha) U_{\mathbf{A}}(x)). \tag{3}$$

It is easy to check that these new operators also fulfill the conditions in Definition 3.5.

Additionally, Hamrawi and Coupland presented a set of axioms for the cardinality of general type-2 fuzzy sets. Nevertheless, they defined the subsethood between two type-2 fuzzy sets by making use of α-plane representations. The following set of axioms is presented in the context of normal and convex functions from a novel perspective.

Definition 3.7. A function $sc : \mathcal{L}^X \to [0,\infty)$ will be called scalar cardinality of T2FSs if for all $\mathcal{A}, \mathcal{B} \in \mathcal{L}^X$, the following properties are satisfied:

C1. If $\mathcal{A}$ is a crisp set (that is $\mu_{\mathcal{A}}(x) \in \left\{\overline{0}, \overline{1}\right\}$ for all $x \in X$) then
$sc(\mathcal{A}) = |Supp(\mathcal{A})|$ where $Supp(\mathcal{A}) = \left\{x \in X \mid \mu_{\mathcal{A}}(x) \neq \overline{0}\right\}$.
C2. If $\mathcal{A} \sqsubseteq \mathcal{B}$ then $sc(\mathcal{A}) \leq sc(\mathcal{B})$.
C3. If $Supp(\mathcal{A}) \cap Supp(\mathcal{B}) = \emptyset$ then:

$$sc(\mathcal{A} \sqcup \mathcal{B}) = sc(\mathcal{A}) + sc(\mathcal{B}).$$

In order to provide some cardinalities for T2FSs we propose then the following extension of membership degree in IVFSs (that is, an interval with lower and upper membership degrees) to a membership degree in T2FSs. Thus, while in IVFSs we use the endpoints of the interval $[L_{\mathbf{A}}(x), U_{\mathbf{A}}(x)]$, we proposed in [40] to modify these endpoints in order to identify the membership degree in T2FS as follows (see Fig. 5):

$$[L_{\mathcal{A}}(x), U_{\mathcal{A}}(x)] \longrightarrow \left[1 - \int_0^1 (\mu_{\mathcal{A}}(x))^L(y)\,dy,\ \int_0^1 (\mu_{\mathcal{A}}(x))^R(y)\,dy\right].$$

In IVFSs, we defined the cardinality of a set using a convex combination of the endpoints of the intervals, which determines the degree to which each interval contributes to the cardinality (see expression (3)). The same approach can be applied to define the cardinality of a T2FSs as follows. Let $\mathcal{A}$ be any type-2 fuzzy set on a finite universe X with $|X| = N$ and normal and convex membership degrees.

Definition 3.8. For each $\alpha \in [0,1]$, we define the operator $|\cdot|_\alpha : \mathcal{L}^X \to [0,\infty)$ as:

$$|\mathcal{A}|_\alpha = \sum_{i=1}^N \left[\alpha\left(1 - \int_0^1 (\mu_{\mathcal{A}}(x_i))^L(y)\,dy\right) + (1-\alpha)\int_0^1 (\mu_{\mathcal{A}}(x_i))^R(y)\,dy\right]. \tag{4}$$

The next result proves that the previous operators are actual cardinalities.

Theorem 3.9. *The operator $|\cdot|_\alpha : \mathcal{L}^X \to [0,\infty)$ given by (4) is a scalar cardinality in T2FSs.*

Proof. C1. If $\mathcal{A}$ is a crisp set, either $\mu_{\mathcal{A}}(x) = \overline{0}$ or $\mu_{\mathcal{A}}(x) = \overline{1}$. Therefore:

$$Supp(\mathcal{A}) = \left\{x \in X \mid \mu_{\mathcal{A}}(x) = \overline{1}\right\}.$$

Note also that:

$$\alpha\left(1-\int_0^1 \overline{0}^L(y)\,dy\right)+(1-\alpha)\int_0^1 \overline{0}^R(y)\,dy=0 \tag{5}$$

and

$$\alpha\left(1-\int_0^1 \overline{1}^L(y)\,dy\right)+(1-\alpha)\int_0^1 \overline{1}^R(y)\,dy=1.$$

Now:

$$\begin{aligned}|\mathcal{A}|_\alpha &= \sum_{x\in Supp(\mathcal{A})}\left[\alpha\left(1-\int_0^1 (\mu_{\mathcal{A}}(x))^L(y)\,dy\right)+(1-\alpha)\int_0^1 (\mu_{\mathcal{A}}(x))^R(y)\,dy\right]\\ &+\sum_{x\notin Supp(\mathcal{A})}\left[\alpha\left(1-\int_0^1 (\mu_{\mathcal{A}}(x))^L(y)\,dy\right)+(1-\alpha)\int_0^1 (\mu_{\mathcal{A}}(x))^R(y)\,dy\right]\\ &=\sum_{x\in Supp(\mathcal{A})} 1 = |Supp(\mathcal{A})|\,.\end{aligned}$$

C2. If $\mathcal{A}\sqsubseteq\mathcal{B}$ we have that $(\mu_{\mathcal{B}}(x))^L(y)\le(\mu_{\mathcal{A}}(x))^L(y)$ and $(\mu_{\mathcal{A}}(x))^R(y)\le(\mu_{\mathcal{B}}(x))^R(y)$ for all $x\in X$ and $y\in[0,1]$. Therefore:

$$1-\int_0^1 (\mu_{\mathcal{A}}(x_i))^L(y)\,dy \le 1-\int_0^1 (\mu_{\mathcal{B}}(x_i))^L(y)\,dy \text{ and}$$

$$\int_0^1 (\mu_{\mathcal{A}}(x_i))^R(y)\,dy \le \int_0^1 (\mu_{\mathcal{B}}(x_i))^R(y)\,dy$$

Hence by expression (4), $|\mathcal{A}|_\alpha\le|\mathcal{B}|_\alpha$.

C3. If $Supp(\mathcal{A})\cap Supp(\mathcal{B})=\emptyset$, for each $x\in X$ there are three possibilities: $\mu_{\mathcal{A}}(x)\ne\overline{0}$ and $\mu_{\mathcal{B}}(x)=\overline{0}$, $\mu_{\mathcal{A}}(x)=\overline{0}$ and $\mu_{\mathcal{B}}(x)\ne\overline{0}$, or $\mu_{\mathcal{A}}(x)=\mu_{\mathcal{B}}(x)=\overline{0}$. Therefore, from Remark 2.16 we have for $x\in Supp(\mathcal{A})$ that $\mu_{\mathcal{A}}(x)\sqcup\mu_{\mathcal{B}}(x)=\mu_{\mathcal{A}}(x)$, for $x\in Supp(\mathcal{B})$ we have $\mu_{\mathcal{A}}(x)\sqcup\mu_{\mathcal{B}}(x)=\mu_{\mathcal{B}}(x)$ and for $x\notin Supp(\mathcal{A})\cup Supp(\mathcal{B})$ we have $\mu_{\mathcal{A}}(x)\sqcup\mu_{\mathcal{B}}(x)=\overline{0}$. Hence, by expression (5):

$$\begin{aligned}|\mathcal{A}\sqcup\mathcal{B}|_\alpha &= \sum_{x\in Supp(\mathcal{A})}\left[\alpha\left(1-\int_0^1 (\mu_{\mathcal{A}}(x_i))^L(y)\,dy\right)+(1-\alpha)\int_0^1 (\mu_{\mathcal{A}}(x_i))^R(y)\,dy\right]\\ &+\sum_{x\in Supp(\mathcal{B})}\left[\alpha\left(1-\int_0^1 (\mu_{\mathcal{B}}(x_i))^L(y)\,dy\right)+(1-\alpha)\int_0^1 (\mu_{\mathcal{B}}(x_i))^R(y)\,dy\right]\\ &=|\mathcal{A}|_\alpha+|\mathcal{B}|_\alpha\,.\quad\square\end{aligned}$$

4. Subsethood in fuzzy and interval-valued fuzzy sets

It is well known that the definition of subsethood for fuzzy sets introduced by Zadeh (see Definition 2.2 in Section 2) is too strong. Accordingly, it may occur that a fuzzy set A is not a subset of another fuzzy set B just because only one membership degree in A is greater than in B, that is, $\mu_A(x)>\mu_B(x)$ for a single element x of the universe X.

Instead, Kosko introduced in [2] the degree of subsethood, or *fuzzy subsethood*, as a measure in fuzzy sets of the degree to which a fuzzy set A is subset of another fuzzy set B. He proposes as such a measure a normalized sum of violations in the membership degree, that is, considering magnitude and proportion of these violations as follows.

Definition 4.1. (Kosko [2]) Let A and B be two fuzzy sets. The degree of subsethood to which A is subset of B, denoted by $S_K(A,B)$, is given by

$$S_K(A,B)=1-\frac{\sum_{x\in X}\max(0,\mu_A(x)-\mu_B(x))}{\sum_{x\in X}\mu_A(x)}.$$

Then, Kosko shows the significance of this degree of subsethood and how it is connected with probability and fuzzy entropy. First, he realizes that the degree of subsethood reduces to *cardinalities*, that is,

$$S_K(A,B) = \frac{|A\cap B|}{|A|}, \tag{6}$$

where $|A|$ is the cardinality of a fuzzy set A given by the sigma-count $|A| = \sum_{x\in X} \mu_A(x)$. Note that, this cardinality is considered only for finite universes of discourse, otherwise the cardinality is infinite. Therefore, this degree of subsethood is also considered only for finite universes.

Furthermore, Kosko also realizes that the degree of subsethood behaves as a conditional probability, that is, *fuzzy conditioning*. Indeed, Zadeh defined in [41] the probability of a fuzzy set A as follows:

$$P(A) = \sum_{x\in X} \mu_A(x)p(x),$$

where $p(x)$ is a probability distribution on a finite universe set X, with $|X| = N$. Therefore, if we assume the uniform distribution of elements in the universe X, that is, $p(x) = \frac{1}{N}$ for all $x \in X$, it holds:

$$S_K(A,B) = \frac{P(A\cap B)}{P(A)} \triangleq P(B|A).$$

Finally, Kosko defines a measure of fuzzy entropy that reduces to fuzzy conditioning, that is, to a subsethood degree measure. He proposes as an entropy measure of a fuzzy set A, denoted by $E_K(A)$, the degree to which the union set between A and its complementary fuzzy set A^c is a subset of the intersection set between A and its complementary A^c, that is,

$$E_K(A) = S_K(A\cup A^c, A\cap A^c). \tag{7}$$

Next, he proves that this is a nonprobabilistic fuzzy entropy since it satisfies the axiomatic properties of entropy proposed by De Luca and Termini for fuzzy sets (see [3]).

Inspired by Kosko's work, Young considered in [5] the following axiomatic properties for a measure of subsethood degree.

Definition 4.2. (Young [5]) Let A and B be two fuzzy sets. A measure of subsethood degree for fuzzy sets, denoted by $S(A,B)$, is a mapping $S: FS(X)\times FS(X) \to [0,1]$ that satisfies the following properties:

S1. $S(A,B) = 1$ if and only if $A \subseteq B$.
S2. If $P \subseteq A$, then $S(A,A^c) = 0$ if and only if $A = X$, being $\mu_P(x) = \frac{1}{2}$ for all $x \in X$.
S3. If $B \subseteq A_1 \subseteq A_2$, then $S(A_2,B) \le S(A_1,B)$, and if $B_1 \subseteq B_2$, then $S(A,B_1) \le S(A,B_2)$.[1]

Later, Young proved that any measure of subsethood degree, $S(A,B)$, that satisfies these axioms provides a measure of fuzzy entropy according to the definition of entropy by Kosko in equation (7), that is, $S(A\cup A^c, A\cap A^c)$ is a fuzzy entropy.

Later, Vlachos and Sergiadis in [8] extended the definition of subsethood degree to IVFS. Similarly to Kosko, Vlachos and Sergiadis considered a normalized sum of violations in the membership degree but for the subsethood of IVFSs. They made use of the order and subsethood relations described in Definitions 2.5 and 2.7, respectively. Accordingly, $\mathbf{A}$ is a subset of $\mathbf{B}$ if and only if for the lower membership degrees it holds $L_{\mathbf{A}} \le L_{\mathbf{B}}$ and for the upper membership degrees it holds $U_{\mathbf{A}} \le U_{\mathbf{B}}$. Thus, the sum of violations in the lower and upper membership degrees, respectively, are given by:

$$\sum_{x\in X} \max(0, L_{\mathbf{A}}(x) - L_{\mathbf{B}}(x)), \quad \sum_{x\in X} \max(0, U_{\mathbf{A}}(x) - U_{\mathbf{B}}(x)).$$

A normalization factor is then $\sum_{x\in X}(L_{\mathbf{A}}(x) + U_{\mathbf{A}}(x))$. Hence, a measure of subsethood degree is:

$$\begin{aligned} S_{IV}(\mathbf{A},\mathbf{B}) &= 1 - \frac{\sum_{x\in X}\left(\max(0, L_{\mathbf{A}}(x) - L_{\mathbf{B}}(x)) + \max(0, U_{\mathbf{A}}(x) - U_{\mathbf{B}}(x))\right)}{\sum_{x\in X}(L_{\mathbf{A}}(x) + U_{\mathbf{A}}(x))} \\ &= \frac{\sum_{x\in X}\left(\min(L_{\mathbf{A}}(x), L_{\mathbf{B}}(x)) + \min(U_{\mathbf{A}}(x), U_{\mathbf{B}}(x))\right)}{\sum_{x\in X}(L_{\mathbf{A}}(x) + U_{\mathbf{A}}(x))}. \end{aligned} \tag{8}$$

Vlachos included a factor of one half in both sums such that it reduces to cardinalities as given in equation (2), and therefore equation (6) also holds for IVFSs. Inspired by Young's work, Vlachos also proposes a set of axiomatic properties that every fuzzy subsethood measure in IVFSs must satisfy.

Definition 4.3. (Vlachos and Sergiadis [8]) Let $\mathbf{A}$ and $\mathbf{B}$ be two interval-valued fuzzy sets. A measure of subsethood degree for IVFS, denoted by $S(\mathbf{A},\mathbf{B})$, is a mapping $S: IVFS(X)\times IVFS(X) \to [0,1]$ that satisfies the following properties:

S1. $S(\mathbf{A},\mathbf{B}) = 1$ if and only if $\mathbf{A} \le_I \mathbf{B}$.
S2. If $\mathbf{A}^c \le_I \mathbf{A}$, then $S(\mathbf{A},\mathbf{A}^c) = 0$ if and only if $\mu_{\mathbf{A}}(x) = [1,1], \forall x \in X$.

[1] Note that, a slightly modified version of this third axiom was later suggested by Fan, Xie and Pei [42].

S3. If $\mathbf{B} \leq_I \mathbf{A_1} \leq_I \mathbf{A_2}$, then $S(\mathbf{A_2}, \mathbf{B}) \leq S(\mathbf{A_1}, \mathbf{B})$, and if $\mathbf{B_1} \leq_I \mathbf{B_2}$, then $S(\mathbf{A}, \mathbf{B_1}) \leq S(\mathbf{A}, \mathbf{B_2})$.

Note that, the second axiom (S2) proposed by Vlachos apparently differs from the one proposed by Young in Definition 4.2. However, the relation $P \subseteq A$ in the sense of Zadeh (that is, for fuzzy sets) is equivalent to $\mathbf{A}^c \leq_I \mathbf{A}$.

Again, Vlachos proves first that $S_{IV}(\mathbf{A}, \mathbf{B})$ given in equation (8) is a measure of fuzzy subsethood in IVFS, and later they also prove that given a fuzzy subsethood measure satisfying the proposed set of axioms given in Definition 4.3, the expression $E_K(\mathbf{A}) = S(\mathbf{A} \cup \mathbf{A}^c, \mathbf{A} \cap \mathbf{A}^c)$ is then a fuzzy entropy measure.

5. Subsethood in type-2 fuzzy sets

This section is devoted to the axiomatic definition of subsethood measures in T2FSs. These new axioms generalize those previously discussed for FSs and IVFSs in Section 4. We also provide some important examples of functions that fulfill such axioms which is the main objective of this work.

Definition 5.1. A mapping $S : T2FS(X) \times T2FS(X) \to [0,1]$ is a subsethood measure in T2FS(X) if it satisfies the following properties:

S1. $S(\mathcal{A}, \mathcal{O}) = 0$ if $\mathcal{A} \neq \mathcal{O}$, being $\mathcal{O}$ the empty type-2 fuzzy set, that is, $\mu_{\mathcal{O}}(x) = \bar{0}$ for all $x \in X$.
S2. $S(\mathcal{A}, \mathcal{B}) = 1$ if and only if $\mathcal{A} \sqsubseteq \mathcal{B}$.
S3. If $\mathcal{B} \sqsubseteq \mathcal{C}$, then $S(\mathcal{A}, \mathcal{B}) \leq S(\mathcal{A}, \mathcal{C})$.
S4. If $\mathcal{A} \sqsubseteq \mathcal{B} \sqsubseteq \mathcal{C}$, then $S(\mathcal{C}, \mathcal{A}) \leq S(\mathcal{B}, \mathcal{A})$.
S5. Let $\mathcal{A}^c \sqsubseteq \mathcal{A}$. Then, $S(\mathcal{A}, \mathcal{A}^c) = 0$ if and only if $\mathcal{A} = \mathcal{X}$, being $\mathcal{X}$ the universal type-2 fuzzy set, that is, $\mu_{\mathcal{X}}(x) = \bar{1}$ for all $x \in X$.

Remark 5.2. The properties S2-S5 where also given by Takáč [10] as the necessary conditions for an operator to be a subsethood measure for T2FSs. Nevertheless, as previously discussed, he employed a different definition for the subsethood of T2FSs. Moreover, Takác did not provide any example of any subsethood measure satisfying the axioms that he defined.

We think it is also reasonable to take into account the boundary condition in S1. This is a consequence of $\bar{0}$ being the smallest element in $\mathcal{L}$.

Our aim now is to define new subsethood measures in terms of cardinality with a similar perspective as Kosko, Vlachos and Sergiadis. To this end, we need to introduce a specific kind of cardinality which we call strict cardinality.

Definition 5.3. A function $ssc : \mathcal{L}^X \to [0, \infty)$ will be called strict scalar cardinality of T2FSs if it is a scalar cardinality in the sense of Definition 3.7 and if for all $\mathcal{A}, \mathcal{B} \in \mathcal{L}^X$, the following property is satisfied:

C2'. If $\mathcal{A} \sqsubset \mathcal{B}$ then $sc(\mathcal{A}) < sc(\mathcal{B})$ where $\mathcal{A} \sqsubset \mathcal{B}$ denotes that $\mathcal{A} \sqsubseteq \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$.

Any strict scalar cardinality can be used to define a subsethood measure in T2FSs satisfying the fuzzy conditioning identified by Kosko in equation (6) as we will check in Theorem 5.5. However, we need first to introduce some preliminary properties regarding the meet operator $\sqcap$ for T2FSs.

Theorem 5.4. *([32], [31]) Let $f, g \in \mathcal{L}$,*

$$(f \sqcap g)(x) = (f(x) \wedge g^R(x)) \vee (f^R(x) \wedge g(x)).$$

From the previous theorem and [31], we can ensure that the operator $\sqcap$ in T2FSs also verifies the following properties:

$$(f \sqcap g)^R(x) = f^R(x) \sqcap g^R(x) = f^R(x) \wedge g^R(x), \tag{9}$$

$$(f \sqcap g)^L(x) = f^L(x) \sqcap g^L(x) = f^L(x) \vee g^L(x). \tag{10}$$

Theorem 5.5. *Let $ssc : \mathcal{L}^X \to [0, \infty)$ be a strict scalar cardinality. The operator $S : \mathcal{L}^X \times \mathcal{L}^X \to [0,1]$ such that:*

$$S(\mathcal{A}, \mathcal{B}) = \frac{ssc(\mathcal{A} \sqcap \mathcal{B})}{ssc(\mathcal{A})} \tag{11}$$

for all $\mathcal{A} \neq \mathcal{O}$ and $S(\mathcal{A}, \mathcal{B}) = 1$ when $\mathcal{A} = \mathcal{O}$, is a subsethood measure.

Proof. Let us show that all the conditions in Definition 5.1 are satisfied by equation (11):

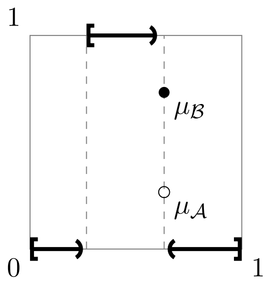

Fig. 6. Example of two type-2 fuzzy sets, $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \sqsubset \mathcal{B}$ and such that $|\mathcal{A}|_\alpha = |\mathcal{B}|_\alpha$ since $\mu_{\mathcal{A}} = \mu_{\mathcal{B}}$ except in one point.

S1. Given $\mathcal{A} \neq \mathcal{O}$, since $\overline{0} \sqsubseteq \mu_{\mathcal{A}}(x)$ for all $x \in X$ we have by Remark 2.16 that $\mu_{\mathcal{A}}(x) \sqcap \overline{0} = \overline{0}$ and $\mu_{\mathcal{A}}(x) \sqcup \overline{0} = \mu_{\mathcal{A}}(x)$. Hence, $\mathcal{A} \sqcap \mathcal{O} = \mathcal{O}$ and $\mathcal{A} \sqcup \mathcal{O} = \mathcal{A}$. Moreover, note that $Supp(\mathcal{O}) = \emptyset$ and then applying the axiom C1 for scalar cardinality $ssc(\mathcal{O}) = |Supp(\mathcal{O})| = 0$. In addition, since ssc is a strict scalar cardinality it is clear that $ssc(\mathcal{A}) \neq 0$. Thus:

$$\mathcal{S}(\mathcal{A}, \mathcal{O}) = \frac{ssc(\mathcal{A} \sqcap \mathcal{O})}{ssc(\mathcal{A})} = \frac{ssc(\mathcal{O})}{ssc(\mathcal{A})} = 0.$$

S2. $\mathcal{S}(\mathcal{A}, \mathcal{B}) = 1$ if and only if $ssc(\mathcal{A} \sqcap \mathcal{B}) = ssc(\mathcal{A})$. Recall that ssc is strict so it is clear that $\mathcal{A} \sqcap \mathcal{B} = \mathcal{A}$ since $\mathcal{A} \sqcap \mathcal{B} \sqsubseteq \mathcal{A}$ (easy to check taking into account Theorem 2.12 and expressions (9) and (10)). If this was not the case we would be facing a contradiction. As a consequence and considering also Remark 2.16, $ssc(\mathcal{A} \sqcap \mathcal{B}) = ssc(\mathcal{A})$ if and only if $\mathcal{A} \sqsubseteq \mathcal{B}$.

S3. If $\mathcal{B} \sqsubseteq \mathcal{C}$, as in the previous point it is easy to check using Theorem 2.12 and expressions (9) and (10) that $\mathcal{A} \sqcap \mathcal{B} \sqsubseteq \mathcal{A} \sqcap \mathcal{C}$ for any $\mathcal{A}$. By the monotonicity of ssc:

$$\mathcal{S}(\mathcal{A}, \mathcal{B}) = \frac{ssc(\mathcal{A} \sqcap \mathcal{B})}{ssc(\mathcal{A})} \leq \frac{ssc(\mathcal{A} \sqcap \mathcal{C})}{ssc(\mathcal{A})} = \mathcal{S}(\mathcal{A}, \mathcal{C}).$$

S4. If $\mathcal{A} \sqsubseteq \mathcal{B} \sqsubseteq \mathcal{C}$, $ssc(\mathcal{B}) \leq ssc(\mathcal{C})$ again by the monotonicity of ssc. Additionally, $\mathcal{C} \sqcap \mathcal{A} = \mathcal{A} = \mathcal{B} \sqcap \mathcal{A}$ using one more time Remark 2.16. Thus:

$$\mathcal{S}(\mathcal{C}, \mathcal{A}) = \frac{ssc(\mathcal{C} \sqcap \mathcal{A})}{ssc(\mathcal{C})} = \frac{ssc(\mathcal{A})}{ssc(\mathcal{C})} \leq \frac{ssc(\mathcal{A})}{ssc(\mathcal{B})} = \frac{ssc(\mathcal{B} \sqcap \mathcal{A})}{ssc(\mathcal{B})} = \mathcal{S}(\mathcal{B}, \mathcal{A}).$$

S5. Let $\mathcal{A}^c \sqsubseteq \mathcal{A}$ or, in other words, $\mathcal{A}^c \sqcap \mathcal{A} = \mathcal{A}^c$. In this situation:

$$0 = \mathcal{S}(\mathcal{A}, \mathcal{A}^c) = \frac{ssc(\mathcal{A} \sqcap \mathcal{A}^c)}{ssc(\mathcal{A})} = \frac{ssc(\mathcal{A}^c)}{ssc(\mathcal{A})}$$

if and only if $ssc(\mathcal{A}^c) = 0 = ssc(\mathcal{O})$. By the strict monotonicity of ssc the previous expressions are equivalent to that of $\mathcal{A}^c = \mathcal{O}$ which is the same as $\mathcal{A} = \mathcal{X}$. □

The search of strict scalar cardinalities for T2FSs is not an easy task since they must be defined over a set of functions. In fact, the cardinalities of Definition 3.8 are not strict cardinalities since there exist some $\mathcal{A} \sqsubset \mathcal{B}$ such that $|\mathcal{A}|_\alpha = |\mathcal{B}|_\alpha$. It should be noted, however, that the only possibility in this case is that of the functions $\mu_{\mathcal{A}}(x)$ and $\mu_{\mathcal{B}}(x)$ differing in a set of measure zero for some $x \in X$. This is deduced from the following remark.

Remark 5.6. Given two T2FSs $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \sqsubset \mathcal{B}$ such that $|\mathcal{A}|_\alpha = |\mathcal{B}|_\alpha$, we have by definition $(\mu_{\mathcal{A}}(x))^L \geq (\mu_{\mathcal{B}}(x))^L$ and $(\mu_{\mathcal{B}}(x))^R \geq (\mu_{\mathcal{A}}(x))^R$. Since $\mathcal{A} \neq \mathcal{B}$, there exist some $x \in X$ such that $(\mu_{\mathcal{A}}(x))^L(y) > (\mu_{\mathcal{B}}(x))^L(y)$ or $(\mu_{\mathcal{B}}(x))^R(y) > (\mu_{\mathcal{A}}(x))^R(y)$ for some values of y. Let us suppose that the set of values fulfilling these conditions has non-zero measure and that $0 < \alpha < 1$. Therefore:

$$|\mathcal{B}|_\alpha - |\mathcal{A}|_\alpha = \sum_{i=1}^{N} \left[\alpha \int_0^1 \left((\mu_{\mathcal{A}}(x_i))^L(y) - (\mu_{\mathcal{B}}(x_i))^L(y) \right) dy \right.$$
$$\left. + (1-\alpha) \int_0^1 \left((\mu_{\mathcal{B}}(x_i))^R(y) - (\mu_{\mathcal{A}}(x_i))^R(y) \right) dy \right] > 0,$$

which is a contradiction. Hence, for each $x \in X$, $\mu_{\mathcal{A}}(x) = \mu_{\mathcal{B}}(x)$ except possibly in a set of measure zero since for each convex function f we have $f = f^L \wedge f^R$. An example of this situation can be found in Fig. 6.

The differences between functions in sets of zero measure is not a significant issue when considering the use of integrals to assess the disparity between such functions. Consequently, we propose the introduction of the following operators as a means of measuring subsethood. Let $\mathcal{A}, \mathcal{B} \in \mathcal{L}^X$ be type-2 fuzzy sets on a finite universe X with $|X| = N$. For each $0 < \alpha < 1$, we define S_α as:

$$S_\alpha(\mathcal{A},\mathcal{B}) = \frac{|\mathcal{A} \sqcap \mathcal{B}|_\alpha}{|\mathcal{A}|_\alpha}$$

for all $\mathcal{A} \neq \mathcal{O}$ and $S_\alpha(\mathcal{O},\mathcal{B}) = 1$. Using now the properties given in equations (9) and (10), and according to the cardinality defined in Definition 3.8, we can establish that:

$$S_\alpha(\mathcal{A},\mathcal{B}) = \frac{\sum_{i=1}^N \alpha\left(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y) \vee \rho^L_{i,\mathcal{B}}(y)\, dy\right) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y) \wedge \rho^R_{i,\mathcal{B}}(y)\, dy}{\sum_{i=1}^N \alpha\left(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y)\, dy\right) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\, dy},$$

where, for readability, $\rho^L_{i,\mathcal{A}}(y) = (\mu_\mathcal{A}(x_i))^L(y)$, $\rho^L_{i,\mathcal{B}}(y) = (\mu_\mathcal{B}(x_i))^L(y)$, $\rho^R_{i,\mathcal{A}}(y) = (\mu_\mathcal{A}(x_i))^R(y)$, and $\rho^R_{i,\mathcal{B}}(y) = (\mu_\mathcal{B}(x_i))^R(y)$.

As we will show in the next theorem the only axiom of subsethood measure in Definition 5.1 that is not fully satisfied is S2. Nevertheless, it is possible to ensure a slightly more relaxed condition. One advantage of this proposal is that it satisfies the fuzzy conditioning identified by Kosko in equation (6).

Theorem 5.7. *Given $0 < \alpha < 1$ and X a finite set with $|X| = N$, the operator $S_\alpha : \mathcal{L}^X \times \mathcal{L}^X \to [0,1]$ such that:*

$$S_\alpha(\mathcal{A},\mathcal{B}) = \frac{|\mathcal{A} \sqcap \mathcal{B}|_\alpha}{|\mathcal{A}|_\alpha}$$

for all $\mathcal{A} \neq \mathcal{O}$ and $S_\alpha(\mathcal{O},\mathcal{B}) = 1$ satisfies the subsethood measure axioms S1,S3-S5 and the condition:

S2'. *$S(\mathcal{A},\mathcal{B}) = 1$ if and only if $\mathcal{A} \sqsubseteq \mathcal{B}$ except possibly in a set of zero measure.*

Proof. First note that S_α is well defined since $|\mathcal{A}|_\alpha \neq 0$ for all $\mathcal{A} \neq \mathcal{O}$. If $\mathcal{A} \neq \mathcal{O}$, there exists $x \in X$ such that $\mu_\mathcal{A}(x) \neq \overline{0}$ and thus there also exists $y_0 \in (0,1]$ with $\mu_\mathcal{A}(x)(y_0) > 0$. In this case $(\mu_\mathcal{A}(x))^R(y) > 0$ for all $y \in [0, y_0]$ and by expression (4) we have $|\mathcal{A}|_\alpha > 0$. This property can be used to prove S1 and S5 in a similar way as for strict cardinalities in Theorem 5.5. Moreover, the proof of S3 and S4 is completely analogous to the one given in the aforementioned theorem. Hence, it only remains to proof S2'.

If $\mathcal{A} = \mathcal{O}$, then by definition it holds $S_\alpha(\mathcal{O},\mathcal{B}) = 1$ and $\mathcal{O} \sqsubseteq \mathcal{B}$ for any $\mathcal{B}$. In any other case, $S_\alpha(\mathcal{A},\mathcal{B}) = 1$ if and only if:

$$\sum_{i=1}^N \left[\alpha\left(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y) \vee \rho^L_{i,\mathcal{B}}(y)\, dy\right) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y) \wedge \rho^R_{i,\mathcal{B}}(y)\, dy\right] = \sum_{i=1}^N \left[\alpha\left(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y)\, dy\right) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\, dy\right]. \quad (12)$$

Taking into account that for all $1 \le i \le N$:

$$\alpha\left(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y) \vee \rho^L_{\mathcal{B}}(y)\, dy\right) \le \alpha\left(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y)\, dy\right)$$

and

$$(1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y) \wedge \rho^R_{\mathcal{B}}(y)\, dy \le (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\, dy,$$

the only possibility for (12) to hold if $0 < \alpha < 1$, is the previous inequalities to be equal. Then, $\rho^L_{\mathcal{B}}(y) \le \rho^L_{i,\mathcal{A}}(y)$ and $\rho^R_{i,\mathcal{A}}(y) \le \rho^R_{\mathcal{B}}(y)$, except possibly in a set of measure zero. Thus we have the condition S2'. □

Fig. 7 shows an example of two normal and convex T2FSs for $|X| = 1$ such that $\sqsubseteq$ fails in a set of measure zero but with the same cardinalities.

Another aspect of S_α that supports its use as a subsethood measure is that it can be seen as the complement of the amount of violations of the usual type-2 subsethood $\sqsubseteq$. In a similar way as Kosko, Vlachos and Sergiadis did for fuzzy sets and interval-valued fuzzy sets, given two convex and normal type-2 fuzzy sets $\mathcal{A}$ and $\mathcal{B}$ we can measure how far are these sets from satisfying $\mathcal{A} \sqsubseteq \mathcal{B}$ by considering the sum of the following integrals:

$$\sum_{i=1}^N \left(\int_0^1 0 \vee \left(\rho^L_{i,\mathcal{B}}(y) - \rho^L_{i,\mathcal{A}}(y)\right) dy + \int_0^1 0 \vee \left(\rho^R_{i,\mathcal{A}}(y) - \rho^R_{i,\mathcal{B}}(y)\right) dy\right)$$

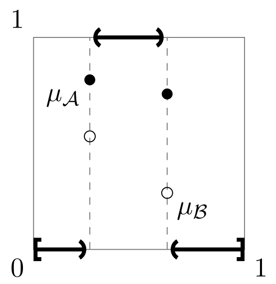

Fig. 7. Example of two type-2 fuzzy sets, $\mathcal{A}$ and $\mathcal{B}$ such that $|\mathcal{A}|_\alpha = |\mathcal{B}|_\alpha = |\mathcal{A} \sqcap \mathcal{B}|_\alpha$ and neither $\mu_\mathcal{A} \sqsubseteq \mu_\mathcal{B}$ nor $\mu_\mathcal{B} \sqsubseteq \mu_\mathcal{A}$.

where $\int_0^1 0 \vee \left(\rho^L_{i,\mathcal{B}}(y) - \rho^L_{i,\mathcal{A}}(y)\right) dy$ and $\int_0^1 0 \vee \left(\rho^R_{i,\mathcal{A}}(y) - \rho^R_{i,\mathcal{B}}(y)\right) dy$ can be interpreted as quantities measuring to what extent the inequalities $\rho^L_{i,\mathcal{B}} \le \rho^L_{i,\mathcal{A}}$ and $\rho^R_{i,\mathcal{A}} \le \rho^R_{i,\mathcal{B}}$ are respectively violated. The previous sum can be normalized by $\sum_{i=1}^N (1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y) + \int_0^1 \rho^R_{i,\mathcal{A}}(y))$ and thus we obtain the subsethood measure:

$$\tilde{S}(\mathcal{A},\mathcal{B}) = 1 - \frac{\sum_{i=1}^N \left(\int_0^1 0 \vee \left(\rho^L_{i,\mathcal{B}}(y) - \rho^L_{i,\mathcal{A}}(y)\right) dy + \int_0^1 0 \vee \left(\rho^R_{i,\mathcal{A}}(y) - \rho^R_{i,\mathcal{B}}(y)\right) dy\right)}{\sum_{i=1}^N \left(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y)\, dy + \int_0^1 \rho^R_{i,\mathcal{A}}(y)\, dy\right)}$$

By the properties of $\vee$, it is easy to check that $\tilde{S} = S_{\frac{1}{2}}$ just by developing the previous expression. By considering S_α with $\alpha \neq 1/2$ we are just giving different weights to the violations of the inequalities $\rho^L_{i,\mathcal{B}} \le \rho^L_{i,\mathcal{A}}$ and $\rho^R_{i,\mathcal{A}} \le \rho^R_{i,\mathcal{B}}$. Consequently, we have a new different justification for the use of the suggested operators as subsethood measures.

Theorem 5.8. *Given $0 < \alpha < 1$ and X a finite set with $|X| = N$, the operator $\overline{S}_\alpha : \mathcal{L}^X \times \mathcal{L}^X \to [0,1]$ such that:*

$$\overline{S}_\alpha(\mathcal{A},\mathcal{B}) = \frac{1}{N} \sum_{i=1}^N \frac{\alpha\left(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y) \vee \rho^L_{i,\mathcal{B}}(y)\, dy\right) + (1-\alpha) \int_0^1 \rho^R_{i,\mathcal{A}}(y) \wedge \rho^R_{i,\mathcal{B}}(y)\, dy}{\alpha\left(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y)\, dy\right) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\, dy} \tag{13}$$

for all $\mathcal{A} \neq \mathcal{O}$ and $\overline{S}_\alpha(\mathcal{O},\mathcal{B}) = 1$ satisfies the subsethood measure axioms S1,S3-S5 and the condition S2'.

Proof. First note that, the function $\overline{S}_\alpha$ is well defined for all $\alpha < 1$ since $\mathcal{A} \neq \mathcal{O}$.

S1. If $\mathcal{A} \neq \mathcal{O}$, then

$$\begin{aligned}\overline{S}_\alpha(\mathcal{A},\mathcal{O}) &= \frac{1}{N}\sum_{i=1}^N \frac{\alpha(1 - \int_0^1 \rho^L_{i,\mathcal{O}}(y))\, dy + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{O}}(y)\, dy}{\alpha(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y)\, dy) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\, dy} \\ &= \frac{1}{N}\sum_{i=1}^N \frac{\alpha(1-1) + (1-\alpha)0}{\alpha(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y)\, dy) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\, dy} = 0.\end{aligned}$$

S2'. If $\mathcal{A} = \mathcal{O}$, then by definition it holds $\overline{S}_\alpha(\mathcal{O},\mathcal{B}) = 1$ and $\mathcal{O} \sqsubseteq \mathcal{B}$ for any $\mathcal{B}$. In any other case, $\overline{S}_\alpha(\mathcal{A},\mathcal{B}) = 1$ if and only if

$$\frac{\alpha(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y) \vee \rho^L_{i,\mathcal{B}}(y)\, dy) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y) \wedge \rho^R_{i,\mathcal{B}}(y)\, dy}{\alpha(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y)\, dy) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\, dy} = 1 \tag{14}$$

for all $1 \le i \le N$ since,

$$0 \le \alpha(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y) \vee \rho^L_{i,\mathcal{B}}(y)\, dy) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y) \wedge \rho^R_{i,\mathcal{B}}(y)\, dy \le$$

$$\alpha(1 - \int_0^1 \rho^L_{i,\mathcal{A}}(y)\, dy) + (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\, dy$$

because for all $1 \le i \le N$:

$$0 \le \alpha\left(1-\int_0^1 \rho^L_{i,\mathcal{A}}(y)\vee\rho^L_{i,\mathcal{B}}(y)\,dy\right) \le \alpha\left(1-\int_0^1 \rho^L_{i,\mathcal{A}}(y)\,dy\right)$$

and

$$0 \le (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\wedge\rho^R_{i,\mathcal{B}}(y)\,dy \le (1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\,dy.$$

Therefore, the only possibility for (14) to hold is the previous inequalities to be equal. Then, $\rho^L_{i,\mathcal{B}}(y) \le \rho^L_{i,\mathcal{A}}(y)$ and $\rho^R_{i,\mathcal{A}}(y) \le \rho^R_{i,\mathcal{B}}(y)$, except possibly in a set of measure zero.

S3. If $\mathcal{B} \sqsubseteq \mathcal{C}$, then it holds $\rho^L_{i,\mathcal{C}} \le \rho^L_{i,\mathcal{B}}$ and $\rho^R_{i,\mathcal{B}} \le \rho^R_{i,\mathcal{C}}$ for all $1 \le i \le N$. Thus,

$$\begin{aligned}\overline{S}_\alpha(\mathcal{A},\mathcal{B}) &= \frac{1}{N}\sum_{i=1}^N \frac{\alpha(1-\int_0^1 \rho^L_{i\mathcal{A}}(y)\vee\rho^L_{i,\mathcal{B}}(y)\,dy)+(1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\wedge\rho^R_{i,\mathcal{B}}(y)\,dy}{\alpha(1-\int_0^1 \rho^L_{i,\mathcal{A}}(y)\,dy)+(1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\,dy}\\ &\le \frac{1}{N}\sum_{i=1}^N \frac{\alpha(1-\int_0^1 \rho^L_{i\mathcal{A}}(y)\vee\rho^L_{i,\mathcal{C}}(y)\,dy)+(1-\alpha)\int_0^1 \rho^R_{i\mathcal{A}}(y)\wedge\rho^R_{i,\mathcal{C}}(y)\,dy}{\alpha(1-\int_0^1 \rho^L_{i,\mathcal{A}}(y)\,dy)+(1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\,dy}\\ &= \overline{S}_\alpha(\mathcal{A},\mathcal{C}).\end{aligned}$$

S4. If $\mathcal{A} \sqsubseteq \mathcal{B} \sqsubseteq \mathcal{C}$, then it holds $\rho^L_{i,\mathcal{C}} \le \rho^L_{i,\mathcal{B}} \le \rho^L_{i,\mathcal{A}}$ and $\rho^R_{i,\mathcal{A}} \le \rho^R_{i,\mathcal{B}} \le \rho^R_{i,\mathcal{C}}$ for all $1 \le i \le N$.

Let us see that $\overline{S}_\alpha(\mathcal{C},\mathcal{A}) \le \overline{S}_\alpha(\mathcal{B},\mathcal{A})$. Applying the previous inequalities,

$$\overline{S}_\alpha(\mathcal{C},\mathcal{A}) = \frac{1}{N}\sum_{i=1}^N \frac{\alpha(1-\int_0^1 \rho^L_{i,\mathcal{A}}(y)\,dy)+(1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\,dy}{\alpha(1-\int_0^1 \rho^L_{i,\mathcal{C}}(y)\,dy)+(1-\alpha)\int_0^1 \rho^R_{i,\mathcal{C}}(y)\,dy}$$

$$\overline{S}_\alpha(\mathcal{B},\mathcal{A}) = \frac{1}{N}\sum_{i=1}^N \frac{\alpha(1-\int_0^1 \rho^L_{i,\mathcal{A}}(y)\,dy)+(1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\,dy}{\alpha(1-\int_0^1 \rho^L_{i,\mathcal{B}}(y)\,dy)+(1-\alpha)\int_0^1 \rho^R_{i,\mathcal{B}}(y)\,dy}$$

In these two equalities, numerators are equal, and the first denominator is greater than the second, thus $\overline{S}_\alpha(\mathcal{C},\mathcal{A}) \le \overline{S}_\alpha(\mathcal{B},\mathcal{A})$.

The only case that has to be considered separately is when $\mathcal{B} = \mathcal{O}$, in such a case, $\overline{S}_\alpha(\mathcal{O},\mathcal{A}) = 1$ and $\overline{S}_\alpha(\mathcal{C},\mathcal{A}) \le \overline{S}_\alpha(\mathcal{O},\mathcal{A})$ trivially.

S5. Let $\mathcal{A}^c \sqsubseteq \mathcal{A}$. Then, $\rho^L_{i,\mathcal{A}} \le \rho^L_{i,\mathcal{A}^c}$ and $\rho^R_{i,\mathcal{A}^c} \le \rho^R_{i,\mathcal{A}}$. Therefore,

$$\overline{S}_\alpha(\mathcal{A},\mathcal{A}^c) = \frac{1}{N}\sum_{i=1}^N \frac{\alpha(1-\int_0^1 \rho^L_{i,\mathcal{A}^c}(y)\,dy)+(1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}^c}(y)\,dy}{\alpha(1-\int_0^1 \rho^L_{i,\mathcal{A}}(y)\,dy)+(1-\alpha)\int_0^1 \rho^R_{i,\mathcal{A}}(y)\,dy} = 0$$

if and only if $\forall i \in \{1,\dots,N\}$ we have $\int_0^1 \rho^L_{i,\mathcal{A}^c}(y)\,dy = 1$ and $\int_0^1 \rho^R_{i,\mathcal{A}^c}(y)\,dy = 0$. That is, if and only if $\mu_{\mathcal{A}^c}(x_i) = \bar{0}, \forall x_i \in X$, if and only if $\mu_{\mathcal{A}}(x_i) = \bar{1}, \forall x_i \in X$. Thus, $\mathcal{A} = X$. □

6. Conclusions

The principal objective of this study has been to extend the concept of cardinality and subsethood measure to the context of T2FSs. We have considered the membership degrees (which are fuzzy sets) as labels for the variable *truth*, as we believe this interpretation is analogous to Zadeh's original definition of T2FSs. In more precise terms, we have proposed novel axiomatic definitions of cardinality and subsethood measure, based on the order provided by Mizumoto and Tanaka in [29,30] which generalizes the most common orders for FSs and IVFSs.

Furthermore, we have introduced new operators that can be used as cardinalities and defined specific subsethood measures in terms of such cardinalities that satisfy a sort of fuzzy conditioning in the sense of Kosko (see [2]). To the best of our knowledge, no specific subsethood measures in T2FSs have been provided previously. Additionally, we have established a general result showing that any strict cardinality can be used to define a subsethood measure for T2FSs.

CRediT authorship contribution statement

Carmen Torres-Blanc: Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Jesus Martinez-Mateo:** Writing – original draft, Methodology, Investigation. **Susana Cubillo:** Writing – review & editing, Writing – original draft, Validation, Supervision, Project administration, Methodology, Investigation, Funding acquisition, Conceptualization. **Luis Magdalena:** Project administration, Investigation, Funding acquisition, Formal analysis. **Francisco Javier Talavera:** Writing – review & editing, Writing – original draft, Validation, Investigation, Formal analysis, Conceptualization. **Jorge Elorza:** Writing – review & editing, Validation, Supervision, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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